ON THE EQUIVARIANT REDUCTION OF STRUCTURE GROUP OF A PRINCIPAL BUNDLE TO A LEVI SUBGROUP

INDRANIL BISWAS AND A. J. PARAMESWARAN

Abstract. Let M be an irreducible projective variety over an algebraically closed field k of characteristic zero equipped with an action of a group Γ. Let E_G be a principal G–bundle over M, where G is a connected reductive algebraic group over k, equipped with a lift of the action of Γ on M. We give conditions for E_G to admit a Γ–equivariant reduction of structure group to H, where $H \subset G$ is a Levi subgroup. We show that for E_G, there is a naturally associated conjugacy class of Levi subgroups of G. Given a Levi subgroup H in this conjugacy class, E_G admits a Γ–equivariant reduction of structure group to H, and furthermore, such a reduction is unique up to an automorphism of E_G that commutes with the action of Γ.

1. Introduction

A holomorphic G–bundle over $\mathbb{CP}^1$ admits a holomorphic reduction of structure group to a maximal torus of G, where G is a complex reductive group [Gr]. In particular, any holomorphic vector bundle over $\mathbb{CP}^1$ splits as a direct sum of line bundles. If V is a holomorphic vector bundle over $\mathbb{CP}^1$ equipped with a lift, as vector bundle automorphisms, of the standard action of the diagonal matrices in $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{CP}^1$, then V decomposes as a direct sum of holomorphic line subbundles each left invariant by the action of the torus [Ku]. More generally, let E_G be a principal G–bundle over an irreducible complex projective variety M, where G is a complex reductive group, with both E_G and M equipped with algebraic actions of a connected complex algebraic group Γ; the action of Γ on E_G commutes with the action of G and the projection of E_G to M is Γ–equivariant. Assume that E_G admits a reduction of structure group to a maximal torus T of G. Then E_G admits a Γ–equivariant reduction of structure group to T if and only if the action of Γ on the automorphism group of E_G leaves a maximal torus invariant [BP].

The aim here is to investigate conditions under which a principal G–bundle over a projective variety equipped with an action of a group Γ admits a Γ–equivariant reduction of structure group to a Levi subgroup of G.

Let M be an irreducible projective variety over an algebraically closed field k of characteristic zero on which a group Γ acts as algebraic automorphisms. Let G be a connected reductive linear algebraic group over k and E_G a principal G–bundle over M. Let $\mathrm{Aut}(E_G)$ denote the group of all automorphisms of E_G. Suppose we are given a lift of the action of Γ on M to E_G that commutes with the action of G. More precisely, the automorphism of

E_G defined by any $\gamma \in \Gamma$ is an algebraic automorphism of G–bundle over the automorphism of M defined by γ. The action of Γ on E_G induces an action on $\text{Aut}(E_G)$ through group automorphisms.

A torus is a product of copies of the multiplicative group $\mathbb{G}_m$. By a Levi subgroup of G we will mean the centralizer of some torus of G. Let $E_H \subset E_G$ be a reduction of structure group to a Levi subgroup H. We will denote by $Z_0(H)$ the connected component of the center of H containing the identity element. So $Z_0(H)$ is contained in the automorphism group of E_H and hence contained in $\text{Aut}(E_G)$.

We prove that E_H is left invariant by the action of Γ on E_G if and only if Γ acts trivially on the subgroup $Z_0(H) \subset \text{Aut}(E_G)$ (Theorem 2.2).

A torus of $\text{Aut}(E_G)$ determines a torus, unique up to inner automorphism, of G. The G–bundle E_G admits a Γ–equivariant reduction of structure group to the Levi subgroup H if and only if there is torus $T \subset \text{Aut}(E_G)$ satisfying the following two conditions:

(1) the action of Γ on T is trivial;
(2) there is a subtorus $T' \subset Z_0(H) \subset G$ in the conjugacy class of tori of G defined by T such that the centralizer of T' in G coincides with H.

(See Lemma 3.2 and Proposition 3.3.)

Let $T_0 \subset G$ be a torus in the conjugacy class of tori defined by a maximal torus of $\text{Aut}(E_G)^\Gamma$, where $\text{Aut}(E_G)^\Gamma \subset \text{Aut}(E_G)$ is the subgroup fixed pointwise by the action of Γ. The conjugacy class of T_0 does not depend on the choice of the maximal torus. Let $H_0 \subset G$ be the centralizer of T_0.

In Theorem 4.1 we prove that E_G admits a Γ–equivariant reduction of structure group $E_{H_0} \subset E_G$ to H_0, which is unique in the following sense:

(1) for any Γ–equivariant reduction of structure group $E'_{H_0} \subset E_G$ to H_0, there is an automorphism $\tau \in \text{Aut}(E_G)^\Gamma$ such that $\tau(E_{H_0}) = E'_{H_0}$ as subvarieties of E_G; and
(2) if E_H is a Γ–equivariant reduction of structure group to a Levi subgroup $H \subset G$, then there is $g \in G$ and $\tau \in \text{Aut}(E_G)^\Gamma$ such that $g^{-1}H_0 g \subset H$ and $E_{H_0} g \subset \tau(E_H)$.

A theorem due to Atiyah says that for an isomorphism of a vector bundle over M with any direct sum of indecomposable vector bundles, the direct summands are unique up to a permutation of the summands. Theorem 4.1 is an equivariant principal bundle analog of this result of [At].

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2. Equivariant reduction to a Levi subgroup

Let M be an irreducible projective variety over an algebraically closed field k of characteristic zero. Let Γ be a group acting on the left on M. So we have a map

$$\phi \,:\, \Gamma\times M \,\longrightarrow\, M \tag{2.1}$$

such that for any $\gamma\,\in\,\Gamma$, the map defined by $x\,\longmapsto\,\phi(\gamma\,,x)$ is an algebraic automorphism of M, and furthermore, $\phi(\gamma_1\,\phi(\gamma_2\,,x))\,=\,\phi(\gamma_1\gamma_2\,,x)$ for all $\gamma_1,\gamma_2\,\in\,\Gamma$ and $x\,\in\,M$, with $\phi(e\,,x)\,=\,x$, where $e\in\Gamma$ is the identity element.

Let G be a connected reductive linear algebraic group over the field k and E_G a principal G–bundle over M. Let

$$f \,:\, E_G \,\longrightarrow\, M \tag{2.2}$$

be the projection. We will denote by $\text{Aut}(E_G)$ the group of all automorphisms of the G-bundle E_G (over the identity automorphism of M). So, $\tau(z)g\,=\,\tau(zg)$ and $f(\tau(z))\,=\,f(z)$ for all $\tau\,\in\,\text{Aut}(E_G)$ and $z\,\in\,E_G$. Note that $\text{Aut}(E_G)$ is an affine algebraic group over k. After fixing a faithful representation of G, the group $\text{Aut}(E_G)$ gets identified with a closed subgroup of the automorphism group of the associated vector bundle; the automorphism group of a vector bundle is a Zariski open dense subset in the vector space defined by the space of all global endomorphisms of the vector bundle. The group $\text{Aut}(E_G)$ is, in fact, the space of all global section of the adjoint bundle

$$\text{Ad}(E_G)\,:=\,E_G\times^G G\,=\,(E_G\times G)/G$$

(the action of any $g\in G$ sends any point $(z\,,g')\,\in\,E_G\times G$ to $(zg\,,g^{-1}g'g)$). Let

$$\text{Aut}^0(E_G)\,\subset\,\text{Aut}(E_G)$$

be the connected component containing the identity element. So $\text{Aut}^0(E_G)$ is a connected affine algebraic group over k.

Assume that E_G is equipped with a lift of the action of Γ on M. So the map

$$\Phi \,:\, \Gamma\times E_G \,\longrightarrow\, E_G \tag{2.3}$$

defining the action has the property that for any $\gamma\,\in\,\Gamma$ the map defined by $z\,\longmapsto\,\Phi(\gamma\,,z)$ is an algebraic automorphism of E_G that commutes with the action of G on E_G, and $f\circ\Phi(\gamma\,,z)\,=\,\phi(\gamma\,,f(z))$, where f is as in (2.2). Note that the action of Γ on E_G induces an action of Γ on $\text{Aut}(E_G)$ through algebraic group automorphisms. More precisely, the action of any $\gamma\,\in\,\Gamma$ sends $F\,\in\,\text{Aut}(E_G)$ to the automorphism defined by

$$z\,\longmapsto\,\Phi(\gamma\,,F(\Phi(\gamma^{-1}\,,z)))\,,$$

where Φ is as in (2.3).

The group Γ acts on the adjoint bundle $\mathrm{Ad}(E_G)$ as follows: the action of any $\gamma \in \Gamma$ sends $(z\,, g) \in E_G \times G$ to

$$(\Phi(\gamma\,, z)\,, g) \in E_G \times G\,; \tag{2.4}$$

this descends to an action of Γ on the quotient $\mathrm{Ad}(E_G) = (E_G \times G)/G$. This descended action of Γ on $\mathrm{Ad}(E_G)$ lifts the action of Γ on M, and it clearly preserves the algebraic group structure of the fibers of $\mathrm{Ad}(E_G)$. The action of Γ on $\mathrm{Ad}(E_G)$ induces an action of Γ on the space of all global sections of $\mathrm{Ad}(E_G)$, namely $\mathrm{Aut}(E_G)$. It is straight–forward to check that this induced action on $\mathrm{Aut}(E_G)$ coincides with the earlier defined action of Γ on $\mathrm{Aut}(E_G)$.

Let

$$\mathrm{Aut}(E_G)^\Gamma \subset \mathrm{Aut}(E_G) \tag{2.5}$$

be the subgroup that is fixed pointwise by the action of Γ on $\mathrm{Aut}(E_G)$. Since the action of each $\gamma \in \Gamma$ is an algebraic automorphism of $\mathrm{Aut}(E_G)$, the subgroup $\mathrm{Aut}(E_G)^\Gamma$ is Zariski closed.

A reduction of structure group of E_G to a closed subgroup $H \subset G$ is a section of E_G/H over M, or equivalently, a closed subvariety $E_H \subset E_G$ closed under the action of H such that the H action on E_H defines a principal H–bundle over M.

Definition 2.1. A reduction of structure group $E_H \subset E_G$ to H is called Γ–*equivariant* if the subvariety E_H is left invariant by the action of Γ on E_G.

Since the actions of Γ and G on E_G commute, there is an induced action of Γ on E_G/H. It is easy to see that E_H is a Γ–equivariant reduction of structure group if and only if the section over M of the bundle $E_G/H \longrightarrow M$ defined by E_H is fixed by the action of Γ on the space of all sections of E_G/H induced by the action on E_G/H.

By a Levi subgroup of G we will mean the centralizer in G of some torus of G. Recall that a torus is a product of copies of $\mathbb{G}_m$ or the trivial group. For a Levi subgroup $H \subset G$, the centralizer in G of the connected component of the center of H containing the identity element coincides with H (see [SS, §3]). If $H \subset G$ is a Levi subgroup, then there is a parabolic subgroup $H \subset P \subset G$ such that H projects isomorphically to the Levi quotient of P. Conversely, if H is a reductive subgroup of a parabolic subgroup $P \subset G$ such that H projects isomorphically to the Levi quotient of P, then H is a Levi subgroup of G. Note that if we take the torus to be the trivial group, then the corresponding Levi subgroup is G itself, and hence in that case the corresponding parabolic subgroup is G.

Take a Levi subgroup $H \subset G$. Let

$$Z_0(H) \subset H$$

be the connected component of the center of H containing the identity element. Let

$$E_H \subset E_G \tag{2.6}$$

be a reduction of structure group of E_G to H. We have

$$Z_0(H) \subset \mathrm{Aut}^0(E_H) \subset \mathrm{Aut}^0(E_G)\,, \tag{2.7}$$

where $\mathrm{Aut}^0(E_H)$ is the connected component of the group of all automorphisms of the H–bundle E_H containing the identity automorphism; the group $Z_0(H)$ acts on E_H as translations (using the action of H on E_H), which makes $Z_0(H)$ a subgroup of $\mathrm{Aut}^0(E_H)$.

Theorem 2.2. *If the reduction E_H in* (2.6) *is Γ–equivariant, then the subgroup $Z_0(H) \subset \mathrm{Aut}^0(E_G)$ in* (2.7) *is contained in $\mathrm{Aut}(E_G)^\Gamma$ (defined in* (2.5)*).*

Conversely, if $Z_0(H) \subset \mathrm{Aut}^0(E_G) \cap \mathrm{Aut}(E_G)^\Gamma$, then the reduction E_H in (2.6) *is Γ–equivariant.*

Proof. Assume that the reduction E_H in (2.6) is Γ–equivariant. For any $\gamma \in \Gamma$, let Φ_γ be the automorphism of the variety E_H defined by the action of γ. The automorphism $g \in Z_0(H) \subset \mathrm{Aut}^0(E_G)$ preserves E_H, and on E_H it coincides with the map $z \longmapsto zg$. Let S_g be the automorphism of the H–bundle E_H defined by $z \longmapsto zg$. Since the actions of G and Γ on E_G commute, we have

$$\Phi_\gamma \circ S_g \circ \Phi_\gamma^{-1} \,=\, S_g \circ \Phi_\gamma \circ \Phi_\gamma^{-1} \,=\, S_g$$

on E_H. Therefore, the two automorphisms, namely $g \in \mathrm{Aut}^0(E_G)$ (in (2.7)) and the image of g by the action γ on $\mathrm{Aut}(E_G)$, coincide over $E_H \subset E_G$. Consequently, these two automorphisms of E_G coincide. In other words, the action of Γ on $\mathrm{Aut}(E_G)$ fixes the subgroup $Z_0(H)$ pointwise. This completes the proof of the first part.

Assume that Γ acts trivially on the subgroup $Z_0(H) \subset \mathrm{Aut}(E_G)$ defined in (2.7). Take a closed point $x \in M$. We will show that the evaluation map

$$f_x \,:\, Z_0(H) \longrightarrow \mathrm{Ad}(E_G)_x \tag{2.8}$$

is injective, where $\mathrm{Ad}(E_G)_x$ is the fiber of $\mathrm{Ad}(E_G)$ over x; the map f_x sends any $s \in Z_0(H)$ to the evaluation at x of the corresponding section (as in (2.7)) of $\mathrm{Ad}(E_G)$.

To prove that f_x is injective, fix a finite dimensional faithful G–module V over k. Let

$$E_V \,:=\, (E_G \times V)/G$$

be the vector bundle over M associated to E_G for the G–module V; the action of any $g \in G$ sends $(z\,,v) \in E_G \times V$ to $(zg\,, g^{-1}v)$. Take any $\sigma \in Z_0(H) \subset \mathrm{Aut}^0(E_G)$. So σ gives an automorphism

$$\sigma' \,\in\, H^0(M,\, \mathrm{Isom}(E_V))$$

of the vector bundle; the automorphism of $E_G \times V$ that sends any $(z\,,v) \in E_G \times V$ to $(\sigma(z)\,,v)$ descends to an automorphism of E_V.

Since M is complete and irreducible, there are no nonconstant functions on it. Therefore, the coefficients of the characteristic polynomial of the endomorphism

$$\sigma'(y) \,\in\, \mathrm{End}((E_V)_y)\,,$$

where $y \in M$ is a closed point, are independent of y. Also, since σ is an element of a torus, namely $Z_0(H)$, the endomorphism $\sigma'(y)$ is semisimple.

If $f_x(\sigma) = \mathrm{Id}_{(E_G)_x}$, where f_x is defined in (2.8), then clearly $\sigma'(x) = \mathrm{Id}_{(E_V)_x}$. Therefore, in that case, all the eigenvalues of $\sigma'(y)$ are 1 for all $y \in M$. Since all $\sigma'(y)$ is semisimple with all eigenvalues 1, it follows immediately that $\sigma'(y)$ is the identity automorphism of $(E_V)_y$ for each $y \in M$.

Since V is a faithful G–module and σ' is the identity automorphism of E_V, we conclude that σ is the identity automorphism of E_G. This proves that the homomorphism f_x defined in (2.8) is injective.

Therefore, using the evaluation map, $M \times Z_0(H) \subset \mathrm{Ad}(E_G)$ is a subgroup–scheme. Since $Z_0(H)$ is preserved by the action of Γ on $\mathrm{Aut}(E_G)$, it follows immediately that the action of Γ on $\mathrm{Ad}(E_G)$ leaves this subgroup–scheme invariant.

Fix an element $g_0 \in Z_0(H)$ such that the Zariski closure of the group generated by g_0 coincides with $Z_0(H)$. Since H is the centralizer of the subgroup $Z_0(H) \subset G$, and the algebraic subgroup generated by g_0 coincides with $Z_0(H)$, we conclude that H coincides with the centralizer of $g_0 \in G$.

Let

$$F : E_G \times G \longrightarrow \mathrm{Ad}(E_G) := (E_G \times G)/G \tag{2.9}$$

be the quotient map. Let

$$\widehat{F} := F^{-1}(\mathrm{image}(\hat{g}_0)) \subset E_G \times G \tag{2.10}$$

be the subvariety, where

$$\hat{g}_0 : M \longrightarrow \mathrm{Ad}(E_G) \tag{2.11}$$

is the section defined by the above element g_0 using the inclusion $Z_0(H) \hookrightarrow \mathrm{Aut}(E_G)$ in (2.7). Set

$$\widehat{E} := \widehat{F} \cap (E_G \times \{g_0\}) \subset E_G \times G\,, \tag{2.12}$$

where $\widehat{F}$ is defined in (2.10), and let

$$E' \subset E_G \tag{2.13}$$

be the image of $\widehat{E}$ (constructed in (2.12)) by the projection of $E_G \times G$ to E_G defined by $(z\,, g) \longmapsto z$.

Since Γ acts trivially on the subgroup $Z_0(H) \hookrightarrow \mathrm{Aut}(E_G)$, the image of the map $\hat{g}_0$ in (2.11) is left invariant by the action of Γ on $\mathrm{Ad}(E_G)$. Since the action of Γ on $\mathrm{Ad}(E_G)$ is the descent, by the projection F in (2.9), of the diagonal action on $E_G \times G$ with Γ acting trivially on G, it follows that E' in (2.13) is left invariant by the action of Γ on E_G.

Since E' is left invariant by Γ, the theorem follows once we show that E' coincides with the subvariety E_H in (2.6).

To prove that $E' = E_H$, first note that

$$E_H \times \{g_0\} \subset \widehat{F} \subset E_G \times G$$

with $\widehat{F}$ defined in (2.10). Indeed, the automorphism of E_H defined by g_0 sends any $z \in E_H$ to zg_0 (since g_0 is in the center of H, this commutes with the action of H and hence it is an automorphism of E_H). This immediately implies that $E_H \times \{g_0\} \subset \widehat{F}$. Consequently, we have $E_H \subset E'$. On the other hand, for any $x \in M$ and $w \in E' \cap (E_G)_x$ it can be shown that the fiber of E' over x is contained in the orbit of w for the action of the centralizer of g_0 in G. Indeed, if $F(w', g') = F(w'g, g')$, where $g, g' \in G$, $w' \in (E_G)_x$ and F as in (2.9), then $gg'g^{-1} = g'$, this being an immediate consequence of the definition of F. Therefore, if $w, wg \in E'$, with $g \in G$, then $g^{-1}g_0g = g_0$.

We already noted that the centralizer of g_0 in G coincides with H. We also saw that $E_H \subset E'$. Therefore, the above observation that any two points of E' over a point $x \in M$ differ by an element of the centralizer of g_0 implies that $E_H = E'$. This completes the proof of the theorem. □

Example 2.3. It may happen that Γ preserves the subgroup $Z_0(H) \subset \mathrm{Aut}^0(E_G)$ in (2.7), but does not preserve $Z_0(H)$ pointwise. We give an example.

Fix a maximal torus $T \subset G$. Take Γ to be the normalizer $N(T)$ of T in G, and equip M with the trivial action of Γ; let G be such that $N(T) \neq T$. Set E_G to be the trivial G–bundle $M \times G$. The group $N(T)$ acts on $M \times G$ as left translations of G. So the induced action of $N(T)$ on $\mathrm{Aut}(E_G) = G$ is the conjugation action. Set $H = T$. The reduction of structure group of E_G to T defined by the inclusion $M \times T \hookrightarrow M \times G$ has the property that the subgroup

$$Z_0(H) = T \subset G = \mathrm{Aut}(E_G)$$

(defined in (2.7)) is left invariant by the action of $N(T)$ (in this case it is the adjoint action of $N(T)$ on G). However no reduction of structure group of E_G to T is left invariant by the action $N(T)$.

The automorphism group of a torus is a discrete group. Therefore, if Γ is a connected algebraic group acting algebraically on E_G, then Γ acts trivially on $Z_0(H)$ provided $Z_0(H)$ is preserved by Γ.

Proposition 2.4. *Let $T \subset \mathrm{Aut}^0(E_G) \cap \mathrm{Aut}^0(E_G)^\Gamma$ be a torus such that there is an element $g \in \mathrm{Aut}^0(E_G)$ satisfying the condition that $g^{-1}Tg = Z_0(H)$, with $Z_0(H)$ constructed in* (2.7) *for the reduction E_H in* (2.6). *Then E_G admits a Γ–equivariant reduction of structure group to the Levi subgroup H.*

Proof. Take T and g as above. So, the image

$$E'_H := g(E_H) \subset E_G$$

is a reduction of structure group of E_G to H, where E_H is the reduction in (2.6). Take any automorphism τ of the principal H–bundle E_H. Using the reduction E_H in (2.6), the automorphism τ gives an automorphism τ_1 of the G–bundle E_G. On the other hand, using the above reduction $E'_H \subset E_G$ together with the isomorphism of E_H with E'_H defined by $z \longmapsto g(z)$ the automorphism τ gives an automorphism τ_2 of E_G. It is easy to see that $\tau_2 = g\tau_1 g^{-1}$.

Therefore, if we substitute E_H by E'_H, then the subgroup $Z_0(H) \subset \mathrm{Aut}^0(E_G)$ in (2.7) gets replaced by $gZ_0(H)g^{-1}$. Now, the second part of Theorem 2.2 says that E'_H is left invariant by the action of Γ on E_G. This completes the proof of the proposition. $\square$

3. Levi reduction from tori in $\mathrm{Aut}(E_G)^\Gamma$

Let $T \subset \mathrm{Aut}^0(E_G)$ be a torus. From the proof of Theorem 2.2 it can be deduced that T determines a torus, unique up to an inner automorphism, in G. This will be explained below with more details.

Fix a point $x \in M$. We saw in the proof of Theorem 2.2 that the evaluation map

$$f_x : T \longrightarrow \mathrm{Ad}(E_G)_x \tag{3.1}$$

is injective. Since $\mathrm{Ad}(E_G) = (E_G \times G)/G$, if we fix a point $z \in (E_G)_x$, then the quotient map F (defined in (2.9)) gives an isomorphism of $\{z\} \times G$ with $\mathrm{Ad}(E_G)_x$. This identification of G with $\mathrm{Ad}(E_G)_x$ constructed using z is an isomorphism of algebraic groups. Furthermore, if we substitute z by zg, $g \in G$, then the corresponding isomorphism of G with $\mathrm{Ad}(E_G)_x$ is the composition of the earlier one with the automorphism of G defined by the conjugation action of g. Therefore, $f_x(T)$, with f_x defined in (3.1), gives a torus in G up to conjugation.

This torus of G, up to conjugation, defined by $f_x(T)$ actually does not depend on the choice of the point x. To prove this, take $z_1, z_2 \in E_G$ with $f(z_i) = x_i$, $i = 1, 2$, where f is as in (2.2). Consider the evaluation homomorphism

$$f_{x_i} : T \longrightarrow \mathrm{Ad}(E_G)_{x_i}$$

which is injective. Let

$$h_{z_i} : T \longrightarrow G \tag{3.2}$$

be the composition of f_{x_i} with the identification of $\mathrm{Ad}(E_G)_{x_i}$ with G defined by z_i. We want to show that the two subgroups, namely image(h_{z_1}) and image(h_{z_2}), of G differ by an inner automorphism of G.

Fix a point $t_0 \in T$ such that the Zariski closed subgroup of T generated by t_0 is T itself. For a finite dimensional G–module V over k, let E_V be the vector bundle associated to E_G for V and $\hat{t}_0$ the automorphism of E_V defined by $t_0 \in \mathrm{Aut}(E_G)$. From the definition of the map h_{z_i} it follows that the automorphism $\hat{t}_0(x_i)$ of $(E_V)_{x_i}$ and the automorphism of

V given by $h_{z_i}(t_0) \in G$ are intertwined by the isomorphism of $(E_V)_{x_i}$ with V constructed using z_i. (Since $E_V = (E_G \times V)/G$, we have an isomorphism of $(E_V)_{x_i}$ with V that sends any $v \in V$ to the image of $(z_i\,, v)$.) We saw in the proof of Theorem 2.2 that the characteristic polynomial of $\hat{t}_0(y) \in \text{Isom}((E_V)_y)$ is independent of y. Therefore, the automorphisms of V defined the two elements $h_{z_1}(t_0)$ and $h_{z_2}(t_0)$ of G have same characteristic polynomial.

On the other hand, if $T'' \subset G$ is a maximal torus, then the algebra of all functions on the affine variety T''/W, where $W := N(T'')/T''$ is the Weyl group with $N(T'')$ the normalizer of T'' in G, is generated by trace function of finite dimensional G–modules over k [St, p. 87, Theorem 2]. Therefore, $h_{z_1}(t_0)$ and $h_{z_2}(t_0)$ differ by an inner automorphism of G (since the characteristic polynomials of $h_{z_1}(t_0)$ and $h_{z_2}(t_0)$ coincide for any G–module). Since image(h_{z_i}) is generated, as a Zariski closed subgroup, by $h_{z_i}(t_0)$, we conclude that the two subgroups image(h_{z_1}) and image(h_{z_2}) differ by an inner automorphism of G.

Remark 3.1. Let $E_H \subset E_G$ be a reduction of structure group to a Levi subgroup $H \subset G$. Consider the torus $Z_0(H) \subset \text{Aut}^0(E_G)$ in (2.7) corresponding to the reduction E_H. By substituting a point of E_H for the point z_i in (3.2) we conclude that the map in (3.2) sends any $g \in Z_0(H) \subset H$ to the point $g \in \text{Aut}^0(E_G)$ (in terms of (2.7)). Consequently, the torus $Z_0(H) \subset G$ is in the conjugacy class of tori given by the torus $Z_0(H) \subset \text{Aut}^0(E_G)$ in (2.7).

We have the following lemma:

Lemma 3.2. *If the G–bundle E_G admits a Γ–equivariant reduction of structure group to a Levi subgroup $H \subset G$, then there is a torus $T \subset \text{Aut}^0(E_G) \cap \text{Aut}(E_G)^\Gamma$ that satisfies the condition that $Z_0(H)$ is the torus in G defined, up to conjugation, by T.*

Proof. Let $E_H \subset E_G$ be a Γ–equivariant reduction of structure group to H. The image of $Z_0(H)$ in $\text{Aut}^0(G)$ by the inclusion map in (2.7) will be denoted by T. The first part of Theorem 2.2 says that $T \subset \text{Aut}(E_G)^\Gamma$.

Fix a point $z \in E_H \subset E_G$. It is easy to see that the torus $h_z(T) \subset G$ coincides with $Z_0(H)$, where h_z is defined as in (3.2) (by composing the evaluation map $T \longrightarrow \text{Ad}(E_G)_{f(z)}$, where f is defined in (2.2), with the isomorphism $\text{Ad}(E_G)_{f(z)} \longrightarrow G$ defined by z). This completes the proof of the lemma. ☐

In the converse direction we have:

Proposition 3.3. *Let $T' \subset G$ be a torus in the conjugacy class of tori determined by a torus $T \subset \text{Aut}^0(E_G)^\Gamma$ and H the centralizer of T' in G. Then E_G admits a Γ–equivariant reduction of structure group to the Levi subgroup H.*

Proof. Fix any point $z \in E_G$ and consider the homomorphism

$$h_z : T \longrightarrow G \tag{3.3}$$

as in (3.2), namely it is the composition of the evaluation map with the identification, constructed using z, of G with $\mathrm{Ad}(E_G)_{f(z)}$, where f is defined in (2.2). There is an element $g \in G$ with $gh_z(T)g^{-1} = T'$, where T' is as in the statement of the proposition.

Let

$$H_z \subset G \tag{3.4}$$

be the centralizer of $h_z(T)$, with h_z defined in (3.3). Since $gh_z(T)g^{-1} = T'$, and the centralizer of $T' \subset G$ is H, we conclude that

$$gH_zg^{-1} = H\,. \tag{3.5}$$

Fix an element $t_0 \in T$ such that the Zariski closure in T of the subgroup generated by t_0 is T itself. As in (2.11), let

$$\hat{t}_0 : M \longrightarrow \mathrm{Ad}(E_G)$$

be the section defined by the automorphism $t_0 \in \mathrm{Aut}(E_G)$. As in (2.10), set

$$\widehat{F} := F^{-1}(\mathrm{image}(\hat{t}_0)) \subset E_G \times G\,,$$

where F is the projection in (2.9). As in (2.12), define

$$\widehat{E} := \widehat{F} \cap (E_G \times \{h_z(t_0)\}) \subset E_G \times G\,,$$

where h_z is defined in (3.3). Let

$$E' \subset E_G \tag{3.6}$$

be the image of $\widehat{E}$ by the projection of $E_G \times G$ to E_G defined by $(y\,,\nu) \longmapsto y$.

We will show that E' constructed in (3.6) is a Γ–equivariant reduction of structure group of E_G to the subgroup H_z defined in (3.4).

For this, we will first show that E' is closed under the action of H_z (for the action of G on E_G). Note that $h_z(t_0)$ is in the center of H_z (as H_z is the centralizer of $h_z(T)$). Therefore, for the projection F in (2.9) we have

$$F(z_1\,, h_z(t_0)) = F(z_1g_1\,, h_z(t_0)) \tag{3.7}$$

for all $z_1 \in E_G$ and and $g_1 \in H_z$. Indeed, the map F clearly has the property that for $g, g' \in G$ and $w' \in E_G$

$$F(w'\,, g') = F(w'g\,, g') \tag{3.8}$$

if and only if $gg'g^{-1} = g'$. From (3.7) it follows immediately that E' is closed under the action of H_z.

It also follows from (3.8) that for any point $y \in M$, the centralizer of $h_z(t_0)$ (in G) acts transitively on the fiber of E' over y. Note that since the Zariski closure of the group

generated by t_0 is T, and H_z is the centralizer (in G) of $h_z(T)$, it follows immediately that the centralizer of $h_z(t_0)$ is H_z.

We still need to show that the fiber of E' over each point $y \in M$ is nonempty. For this note that there is a point $z' \in f^{-1}(y)$, with f defined in (2.2), such that the corresponding homomorphism

$$h_{z'} : T \longrightarrow G$$

defined as in (3.3) by replacing z by z' has the property that $h_{z'}(t_0) = h_z(t_0)$. Indeed, this follows from the combination of the fact that the conjugacy class of the torus $h_z(T) \subset G$ is independent of the choice of the point $z \in E_G$ and the observation that the two homomorphisms h_{z_1} and $h_{z_1 g_1}$ from T to G, where $z_1 \in E_G$ and $g_1 \in G$, differ by the inner automorphism of G defined by g_1. The identity $h_{z'}(t_0) = h_z(t_0)$ immediately implies that z' is in the fiber of E' over y.

Consequently, $E' \subset E_G$ constructed in (3.6) is a reduction of structure group to H_z.

Since the action of Γ on $\mathrm{Aut}(E_G)$ fixes t_0, the action of Γ on E_G leaves E' invariant.

Finally, from (3.5) it follows immediately that $E'g^{-1} \subset E_G$ is a reduction of structure group to H. As E' is left invariant by the action of Γ on E_G, and the actions of Γ and G on E_G commute, the subvariety $E'g^{-1} \subset E_G$ is also left invariant by the action of Γ. This completes the proof of the proposition. $\square$

4. A canonical equivariant Levi reduction

Let $T \subset \mathrm{Aut}(E_G)^\Gamma$ be a connected maximal torus, where $\mathrm{Aut}(E_G)^\Gamma$ is defined in (2.5). So T is a torus of $\mathrm{Aut}^0(E_G)$. We saw in the previous section that T determines a torus, unique up to an inner conjugation, in G. We will show that this torus in G (up to conjugation) does not depend on the choice of the maximal torus T.

To prove this, first note that any two maximal tori of $\mathrm{Aut}(E_G)^\Gamma$ differ by an inner automorphism of $\mathrm{Aut}(E_G)^\Gamma$. Consider the maximal torus $g_0Tg_0^{-1}$, where $g_0 \in \mathrm{Aut}(E_G)^\Gamma$, and fix a point $z \in E_G$. The point z defines two injective homomorphisms

$$h_z : T \longrightarrow G$$

and

$$h'_z : g_0Tg_0^{-1} \longrightarrow G$$

defined as in (3.2) using the evaluation map and the isomorphism of groups

$$\phi_z : \mathrm{Ad}(E_G)_{f(z)} \longrightarrow G$$

constructed using z, where f is defined in (2.2). From the construction of h_z and h'_z it follows immediately that

$$\phi_z(g_0(f(z)))h_z(T)(\phi_z(g_0(f(z))))^{-1} = h'_z(g_0Tg_0^{-1}) .$$

Therefore, $h_z(T)$ and $h'_z(g_0 T g_0^{-1})$ differ by an inner automorphism of G. Consequently, the torus of G determined by a maximal torus of $\mathrm{Aut}(E_G)^\Gamma$ does not depend on the choice of the maximal torus.

Fix a torus $T_0 \subset G$ in the conjugacy class of tori given by a maximal torus in $\mathrm{Aut}(E_G)^\Gamma$. The centralizer of T_0 in G is a Levi subgroup. This Levi subgroup of G will be denoted by H_0.

Theorem 4.1. *The G–bundle E_G admits a Γ–equivariant reduction of structure group to the Levi subgroup H_0 defined above.*

If $H \subsetneq H_0$ is a Levi subgroup of G properly contained in H_0, then E_G does not admit any Γ–equivariant reduction of structure group to H.

If $H \subset G$ is a Levi subgroup such that E_G admits a Γ–equivariant reduction of structure group to H, but E_G does not admit a Γ–equivariant reduction of structure group to any Levi subgroup properly contained in H, then H is conjugate to the above defined subgroup $H_0 \subset G$.

If $E_{H_0} \subset E_G$ and $E'_{H_0} \subset E_G$ are two Γ–equivariant reductions of structure group to H_0, then there is an automorphism $\tau \in \mathrm{Aut}(E_G)^\Gamma$ of E_G such that $\tau(E_{H_0}) = E'_{H_0} \subset E_G$.

Proof. That E_G admits a Γ–equivariant reduction of structure group to H_0 follows from the construction in Proposition 3.3. Fix a maximal torus $T \subset \mathrm{Aut}^0(E_G)^\Gamma$ and a point $t_0 \in T$ such that the Zariski closure of the subgroup of T generated by t_0 coincides with T. Let $t'_0 \in T_0$ be the element corresponding to t_0 by an isomorphism of T with T_0 constructed using an element of E_G. As in (2.12), consider

$$\widehat{E} := F^{-1}(\mathrm{image}(\hat{t}_0)) \cap (E_G \times \{t'_0\}) \subset E_G \times G\,,$$

where F is defined in (2.9) and $\hat{t}_0$ is the section of $\mathrm{Ad}(E_G)$ defined by t_0. Finally the image of $\widehat{E}$ by the projection of $E_G \times G$ to E_G gives a reduction of structure group of E_G to H_0. See the proof of Proposition 3.3 for the details.

To prove the second statement, let $H \subsetneq H_0$ be a Levi subgroup of G properly contained in H_0. So $\dim Z_0(H) > \dim T_0$, where $Z_0(H)$ is the connected component of the center of H containing the identity element (note that T_0 is contained in the center of the bigger Levi subgroup). The first statement in Theorem 2.2 says that if $E_H \subset E_G$ is a Γ–equivariant reduction of structure group to H, then $\mathrm{Aut}(E_G)^\Gamma$ contains a torus isomorphic to $Z_0(H)$. This is impossible, since a smaller dimensional torus, namely T_0, is isomorphic to the maximal torus T and any two maximal tori are isomorphic.

Let $H \subset G$ be a Levi subgroup as in the third statement, and let $E_H \subset E_G$ be a Γ–equivariant reduction of structure group to H. The condition on H implies that the torus $Z_0(H) \subset \mathrm{Aut}(E_G)^\Gamma$ in (2.7) for the reduction E_H is a maximal torus of $\mathrm{Aut}(E_G)^\Gamma$. Indeed, that $Z_0(H) \subset \mathrm{Aut}(E_G)^\Gamma$ follows from Theorem 2.2. That $Z_0(H)$ is a maximal torus of

$\text{Aut}(E_G)^\Gamma$ can be seen as follows. If $T'' \subset \text{Aut}(E_G)^\Gamma$ is a torus with $Z_0(H) \subsetneq T''$, then take a torus T_1'' in the conjugacy class of tori of G given by T'' such that $Z_0(H) \subset T_1'' \subset G$. Let $H'' \subset G$ be the centralizer of T_1''. Since $Z_0(H)$ is the connected component of the center of H containing the identity element and $Z_0(H) \subsetneq T_1''$ is a proper subtorus, we conclude that $H'' \subsetneq H$. Proposition 3.3 says that E_G admits a Γ–equivariant reduction of structure group to H''. Since H'' is a Levi subgroup properly contained in H, this contradicts the given condition on H. Therefore, $Z_0(H) \subset \text{Aut}(E_G)^\Gamma$ is a maximal torus.

Since T_0, by definition, is in the conjugacy class of tori of G given by a maximal torus of $\text{Aut}(E_G)^\Gamma$, using Remark 3.1 we conclude that the two tori T_0 and $Z_0(H)$ of G are conjugate. Consequently, H and H_0 differ by an inner automorphism of G.

Let E_{H_0} and E'_{H_0} be as in the fourth statement. Consider the inclusion in (2.7). Let T_1 (respectively, T_1') be the image of T_0 in $\text{Aut}(E_G)^\Gamma$ for the reduction E_{H_0} (respectively, E'_{H_0}) by (2.7). From dimension consideration we know that both T_1 and T_1' are maximal tori in $\text{Aut}(E_G)^\Gamma$. Take an element $\tau \in \text{Aut}(E_G)^\Gamma$ such that

$$T_1' = \tau^{-1} T_1 \tau \,. \tag{4.1}$$

Let $E_H \subset E_G$ be a Γ–equivariant reduction of structure group to a Levi subgroup $H \subset G$ and $g_0 \in Z_0(H)$ an element in the connected component of the center of H containing the identity element such that g_0 generates $Z_0(H)$ as a Zariski closed subgroup of G. In the proof of Theorem 2.2 we gave a reconstruction of E_H from g_0 and its image in $\text{Aut}^0(E_G)$ by (2.7). (In the notation of the proof of Theorem 2.2, $E' \subset E_G$ was constructed in (2.13) using g_0 and its image in $\text{Aut}^0(E_G)$, and it was shown there that E_H coincides with E'.)

Fix an element $g_0 \in T_0$ such that Zariski closed subgroup generated by g_0 coincides with T_0. Let g_1 be the image of g_0 in T_1 for the above isomorphism of T_0 with T_1 constructed using E_H. Set

$$g_1' = \tau^{-1} g_1 \tau \in T_1' ,$$

where τ is as in (4.1). Let g_0' be the image of g_1' in T_0 for the above isomorphism of T_1' with T_0 constructed using E'_H. Following the construction of E' in (2.13), we can reconstruct E_H (respectively, E'_H) using the pair $(g_0\,, g_1)$ (respectively, $(g_0'\,, g_1')$). Using this reconstruction it is easy to see that

$$E'_H = \tau^{-1}(E_H) ,$$

where τ is as in (4.1). This completes the proof of the theorem. $\square$

Remark 4.2. If we set $G = \text{GL}(n, k)$ and $\Gamma = \{e\}$, then Theorem 4.1 becomes the following theorem proved in [At]: any vector bundle V over M is isomorphic to a direct

sum $\bigoplus_{i=1}^k U_i$ of indecomposable vector bundles, and if

$$V \cong \bigoplus_{j=1}^{l} W_j\, ,$$

where each W_j is indecomposable, then $k = l$ and the collection of vector bundles $\{W_j\}$ is a permutation of $\{U_i\}$.

Remark 4.3. Let E_* be a parabolic G–bundle over an irreducible smooth projective variety X. Corresponding to E_*, there is an irreducible smooth projective variety Y, a finite subgroup $\Gamma \subset \mathrm{Aut}(Y)$ with $X = Y/\Gamma$, and a principal G–bundle E_G over Y equipped with a lift of the action of Γ. More precisely, there is a bijective correspondence between parabolic G–bundles and G–bundles with a finite group action on a (ramified) covering (see [BBN] for the details). Therefore, Theorem 4.1 gives a natural reduction of structure group of a parabolic G–bundle to a Levi subgroup of G. This Levi reduction satisfies all the analogous conditions in Theorem 4.1.

5. The Levi quotient of the automorphism group

In this final section, we will assume Γ to be a connected algebraic group. We will also assume the action of Γ on E_G to be algebraic, that is, the map ϕ in (2.1) is algebraic; consequently, the action of Γ on M is also algebraic. Since Γ is connected, the action of Γ on $\mathrm{Aut}(E_G)$ preserves the subgroup $\mathrm{Aut}^0(E_G)$.

Let $U\mathrm{Aut}^0(E_G)$ be the unipotent radical of the algebraic group $\mathrm{Aut}^0(E_G)$ [Hu, p. 125]. So the Levi quotient

$$L\mathrm{Aut}^0(E_G) := \mathrm{Aut}^0(E_G)/U\mathrm{Aut}^0(E_G) \tag{5.1}$$

is a connected reductive algebraic group over k. Let

$$\psi : \mathrm{Aut}^0(E_G) \longrightarrow L\mathrm{Aut}^0(E_G) \tag{5.2}$$

be the quotient map.

From the uniqueness of a unipotent radical it follows immediately that the action of Γ on $\mathrm{Aut}^0(E_G)$ preserves the subgroup $U\mathrm{Aut}^0(E_G)$. Therefore, we have an induced action of Γ on $L\mathrm{Aut}^0(E_G)$.

Let $\widehat{T}_0 \subset G$ be a torus in the conjugacy class of tori of G given by a maximal torus in $\mathrm{Aut}^0(E_G)$. Since any two maximal tori are conjugate, the conjugacy class of $\widehat{T}_0$ does not depend on the choice of the maximal torus. Let $\widehat{H}_0$ be the centralizer of T_0 in G. Setting $\Gamma = \{e\}$ in Proposition we conclude that E_G admits a reduction of structure group to $\widehat{H}_0$.

Proposition 5.1. *If E_G admits a Γ–equivariant reduction of structure group to the Levi subgroup $\widehat{H}_0$, then the induced action of Γ on $L\mathrm{Aut}^0(E_G)$ (defined in (5.1)) factors through an action of a torus quotient of Γ.*

If Γ is reductive and the induced action of Γ on $L\mathrm{Aut}^0(E_G)$ factors through an action of a torus quotient of Γ, then E_G admits a Γ–equivariant reduction of structure group to the Levi subgroup $\widehat{H}_0$.

Proof. Assume that E_G admits a Γ–equivariant reduction of structure group to $\widehat{H}_0$. Theorem 4.1 says that there is a maximal torus

$$T_0 \subset \mathrm{Aut}^0(E_G)$$

which is left invariant by the action of Γ on $\mathrm{Aut}(E_G)$. Consider $\psi(T_0)$, with ψ defined in (5.2), which is a maximal torus in $L\mathrm{Aut}^0(E_G)$. Note that $\psi(T_0)$ is left invariant by the induced action of Γ on $L\mathrm{Aut}^0(E_G)$, as T_0 is Γ–invariant.

Let $ZL\mathrm{Aut}^0(E_G) \subset L\mathrm{Aut}^0(E_G)$ be the center and

$$PL\mathrm{Aut}^0(E_G) := L\mathrm{Aut}^0(E_G)/ZL\mathrm{Aut}^0(E_G)$$

the corresponding adjoint group. All the automorphisms of $L\mathrm{Aut}^0(E_G)$ connected to the identity automorphism are parametrized by $PL\mathrm{Aut}^0(E_G)$, with $PL\mathrm{Aut}^0(E_G)$ acting on $L\mathrm{Aut}^0(E_G)$ as conjugations.

Since Γ is connected, we have a homomorphism of algebraic groups

$$\rho : \Gamma \longrightarrow PL\mathrm{Aut}^0(E_G)$$

such that the action of any $g \in \Gamma$ on $L\mathrm{Aut}^0(E_G)$ is conjugation by $\rho(g)$. Since the action of Γ preserves the maximal torus $\psi(T_0) \subset L\mathrm{Aut}^0(E_G)$, and $q \circ \psi(T_0)$ is a maximal torus in $PL\mathrm{Aut}^0(E_G)$, where

$$q : L\mathrm{Aut}^0(E_G) \longrightarrow PL\mathrm{Aut}^0(E_G)$$

is the projection, we conclude that $\rho(\Gamma) \subset q \circ \psi(T_0)$, where ψ is defined in (5.2). (The maximal torus $q \circ \psi(T_0)$ is a finite index subgroup of its normalizer in $PL\mathrm{Aut}^0(E_G)$.) Therefore, the action of Γ on $L\mathrm{Aut}^0(E_G)$ factors through the conjugation action of the torus $\rho(\Gamma)$.

To prove the second statement in the proposition, assume that the induced action of Γ on $L\mathrm{Aut}^0(E_G)$ factors through the torus quotient $\Gamma \longrightarrow T_\Gamma$. We will first show that the action of T_Γ on $L\mathrm{Aut}^0(E_G)$ preserves a maximal torus.

Construct the semi-direct product $L\mathrm{Aut}^0(E_G) \rtimes T_\Gamma$ using the induced action of T_Γ on $L\mathrm{Aut}^0(E_G)$. We take a maximal torus

$$\widehat{T} \subset L\mathrm{Aut}^0(E_G) \rtimes T_\Gamma$$

containing T_Γ (note that T_Γ is naturally a subgroup of $L\mathrm{Aut}^0(E_G) \rtimes T_\Gamma$). Finally, consider the intersection

$$T_1 := \widehat{T} \cap L\mathrm{Aut}^0(E_G)$$

(note that $L\mathrm{Aut}^0(E_G)$ is a normal subgroup of $L\mathrm{Aut}^0(E_G) \rtimes T_\Gamma$). From its construction it is immediate that T_1 is a maximal torus of $L\mathrm{Aut}^0(E_G)$ and T_1 is left invariant by the action of T_Γ on $L\mathrm{Aut}^0(E_G)$.

Consider

$$G' := \psi^{-1}(T_1) \subset \mathrm{Aut}^0(E_G),$$

where ψ is the projection in (5.2). Since Γ preserves $T_1 \subset L\mathrm{Aut}^0(E_G)$, the action of Γ on $\mathrm{Aut}^0(E_G)$ preserves the subgroup G' defined above. Note that G' fits in an exact sequence

$$e \longrightarrow U\mathrm{Aut}^0(E_G) \longrightarrow G' \longrightarrow T_1 \longrightarrow e,$$

where $U\mathrm{Aut}^0(E_G)$, as before, is the unipotent radical.

A maximal torus of G' is a maximal torus of $\mathrm{Aut}^0(E_G)$, and since Γ is connected, an algebraic action of Γ on a torus through automorphisms is trivial. Therefore, in view of Proposition 3.3, to prove the second statement in the proposition it suffices to show that Γ preserves some maximal torus in G'.

Denote by $\mathfrak{g}'$ the Lie algebra of G'. The action of Γ on G' induces an action of Γ on $\mathfrak{g}'$. Let $\mathfrak{u}$ (respectively, $\mathfrak{t}_1$) be the Lie algebra of $U\mathrm{Aut}^0(E_G)$ (respectively, T_1). So the above exact sequence of groups give an exact sequence

$$0 \longrightarrow \mathfrak{u} \longrightarrow \mathfrak{g}' \xrightarrow{\beta} \mathfrak{t}_1 \longrightarrow 0 \tag{5.3}$$

of Lie algebras.

Let

$$\mathcal{V} \subset \mathfrak{g}'$$

be the subspace on which Γ acts trivially. Note that $\mathcal{V}$ is a Lie subalgebra. The action of Γ on T_1 is trivial (as the automorphism group of T_1 is discrete and Γ is connected). Therefore, the induced action of Γ on $\mathfrak{t}_1$ is trivial.

Since Γ is reductive, any exact sequence of finite dimensional Γ–modules over k splits, in particular, (5.3) splits. Since $\mathfrak{t}_1$ is the trivial Γ–module, we conclude that the restriction to the subalgebra $\mathcal{V} \subset \mathfrak{g}'$ of the projection β in (5.3) is surjective.

Let $G_2 \subset G'$ be the Zariski closed subgroup generated by the subalgebra $\mathcal{V}$. Since Γ acts trivially on $\mathcal{V}$ we conclude that G_2 is fixed pointwise by the action of Γ on G'.

Since the projection of $\mathcal{V}$ to $\mathfrak{t}_1$ (by β in (5.3)) is surjective, the subgroup G_2 projects surjectively to T_1. Take any maximal torus $T_2 \subset G_2$. Since the projection of G_2 to T_1 is surjective and the kernel of the projection $G' \longrightarrow T_1$ is a unipotent group, we conclude that T_2 is a maximal torus of G'.

In other words, T_2 is a Γ–invariant maximal torus of G'. Since a maximal torus in G' is a maximal torus in $\mathrm{Aut}^0(E_G)$, Proposition 3.3 completes the proof of the proposition. $\square$

It is easy to construct examples showing that the second statement in Proposition 5.1 is not valid for arbitrary connected algebraic group Γ.

Proposition 5.1 has the following corollary:

Corollary 5.2. *If Γ does not have a nontrivial torus quotient (for example, if it is unipotent or semisimple), and the action of Γ on $L\mathrm{Aut}^0(E_G)$ is nontrivial, then E_G does not admit any Γ–equivariant reduction of structure group to $\widehat{H}_0$, provided $\widehat{H}_0 \neq G$.*

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School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

E-mail address: indranil@math.tifr.res.in

E-mail address: param@math.tifr.res.in